Bilateral Trade with Interdependent Values

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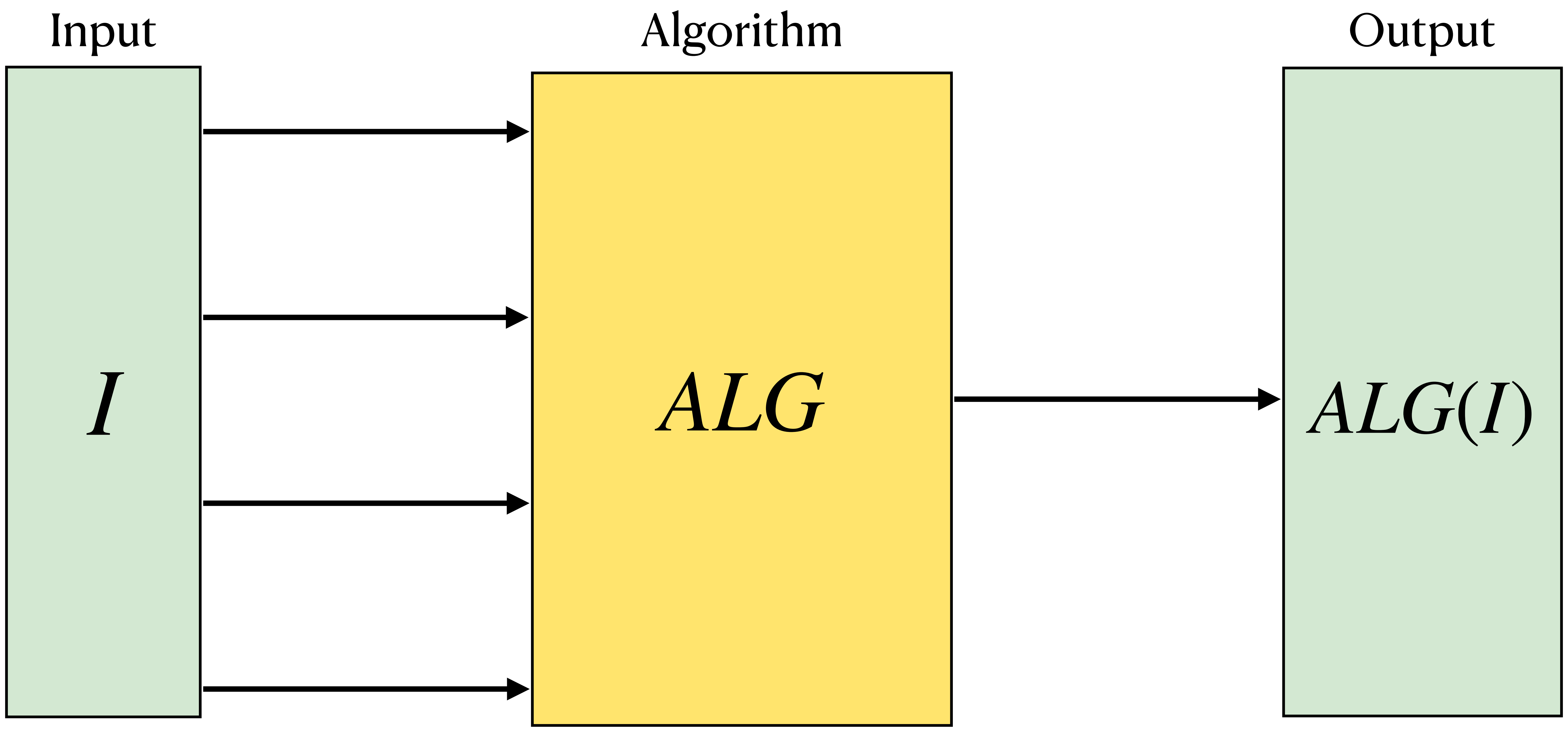
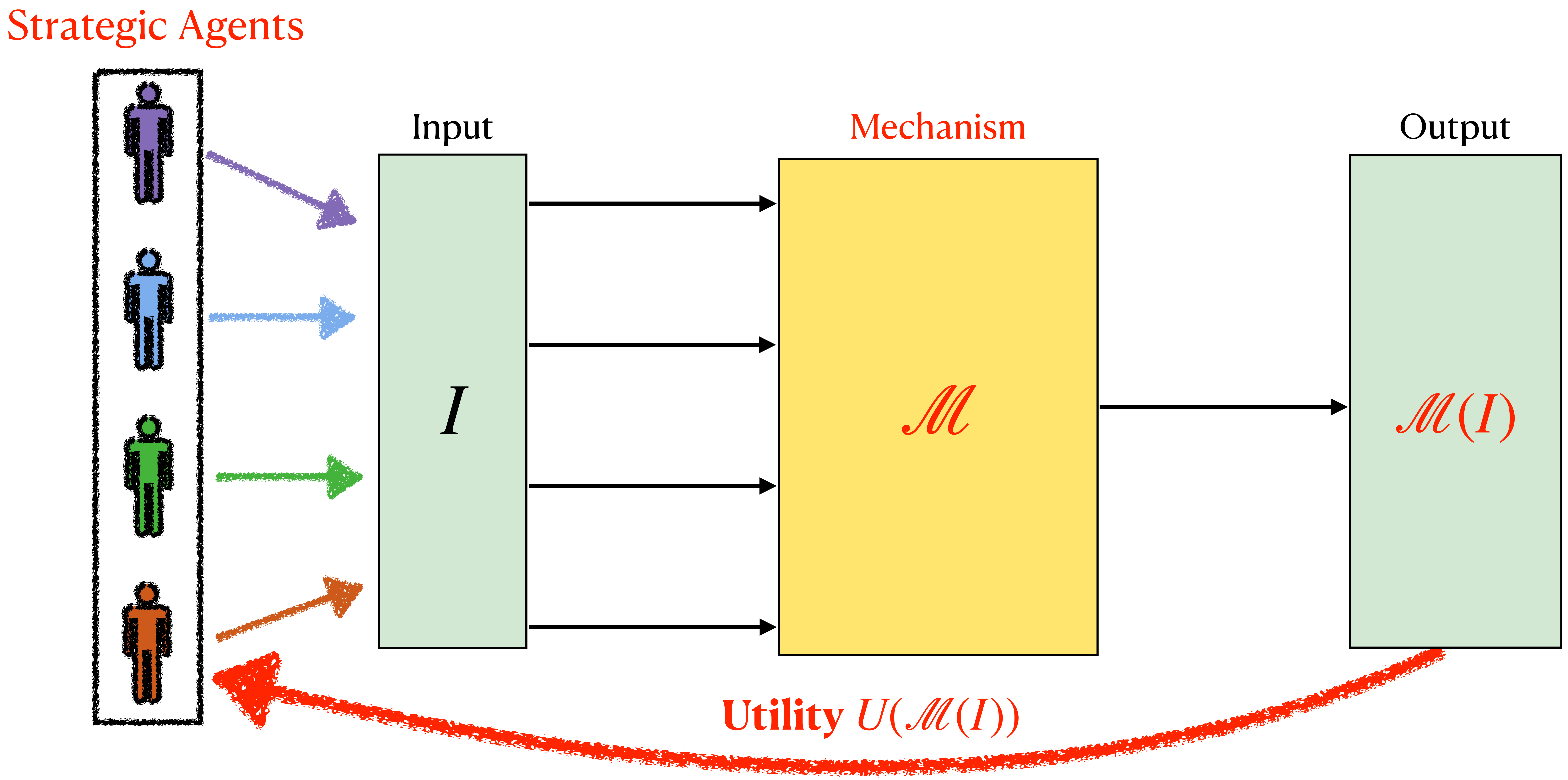
Algorithm Design 101



***Goal**: Optimize over some objective (e.g. maximize efficiency, minimize cost)*

Mechanism Design 101



Goal: *Optimize over some objective (e.g. maximize efficiency, minimize cost)*

Added Concern: *Make sure that the agents report their true information.*

Bilateral trade

Setting: A seller with one item and a potential buyer. The seller values the item v_s and the buyer values it v_b, and these values are **private information**, drawn from **publicly** known distributions.

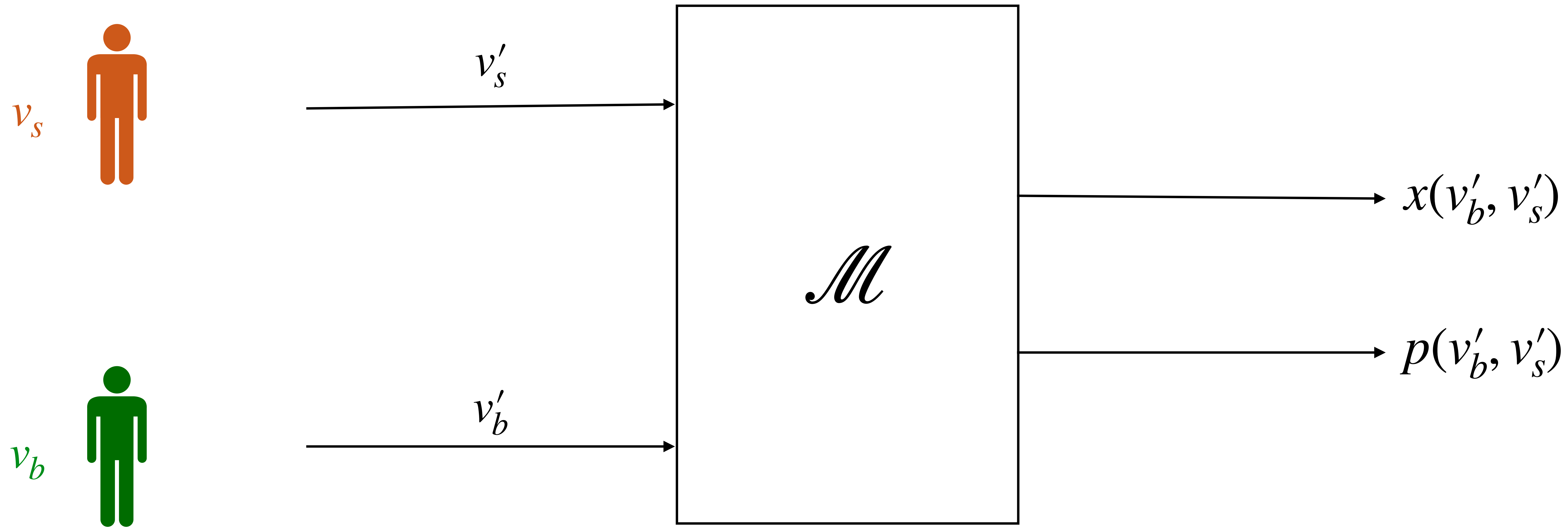
Designer task: Decide if the trade should happen and under what payment scheme.

Natural goal: Trade whenever the buyer values the item more than the seller ($v_b > v_s$).

Mechanism Design Task

A (direct) **mechanism** $\mathcal{M}$ consists of two functions $\mathcal{M} = (x, p)$, takes **reported** values v_b' and v_s' as input and outputs:

1. The **probability** of trade $x(v_b', v_s')$.
2. The **price** that the buyer pays to the seller $p(v_b', v_s')$.



Utilities & Mechanism Constraints

Utilities under mechanism $\mathscr{M}$ with reports (v_b', v_s'):

Seller utility: $p(v_s', v_b') - v_s \cdot x(v_s', v_b')$ Buyer Utility: $v_b \cdot x(v_s', v_b') - p(v_s', v_b')$

Desired Mechanism Constraints:

1. **Individual Rationality** (IR) -----> Non-negative utility from participating in the mechanism.
2. **Incentive Compatibility** (IC) --> No incentive to misreport my information to the mechanism.
3. **Budget Balance** (BB) ------------> The designer does not subsidize the trade.

Objectives

A mechanism's **performance** in bilateral trade is commonly measured in:

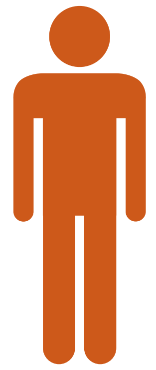
1. Social Welfare: the **value** (welfare) of the agent that is allocated the item.

   The benchmark to compare against is $SW = \mathbb{E}\left[max(v_b, v_s)\right]$.

2. Gains from Trade: the **welfare increase** due to the trade (if it happens).

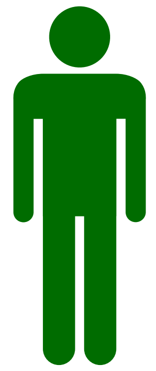
   The benchmark to compare against is $GFT = \mathbb{E}\left[max(v_b - v_s, 0)\right]$.

An Illustrative Example



$v_s = 2.$

Trade?



$$v_b = \begin{cases} 4, & \text{w.p. } \frac{1}{2}, \\ 0, & \text{otherwise}. \end{cases}$$

Distribution

At Optimality:

- Optimal Welfare:

$$SW = \mathbb{E}\left[max(v_b, v_s)\right] = 3.$$

- Optimal GFT:

$$GFT = \mathbb{E}\left[max(v_b - v_s, 0)\right] = 1.$$

Optimal Objectives

Example mechanism:

No **trade** mechanism: $(x, p) = (\mathbf{0}, \mathbf{0})$.

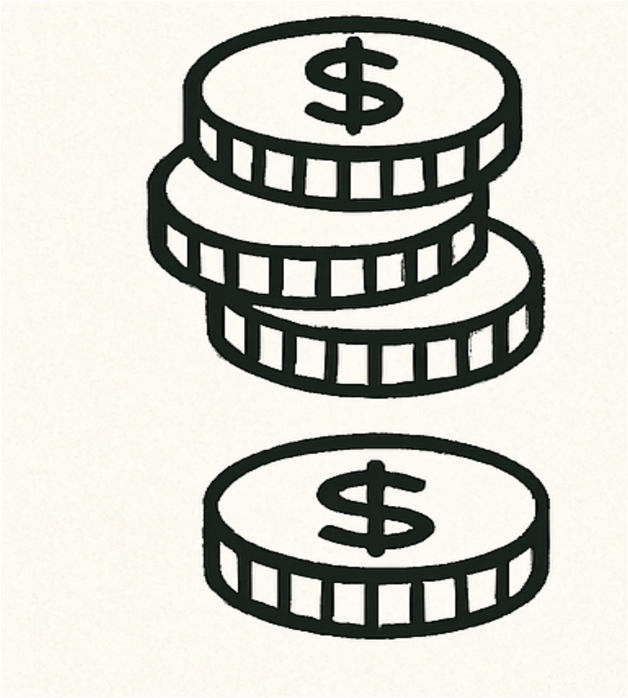
- Expected Welfare:

$$\mathbb{E}[Welfare] = 2.$$

- Expected GFT:

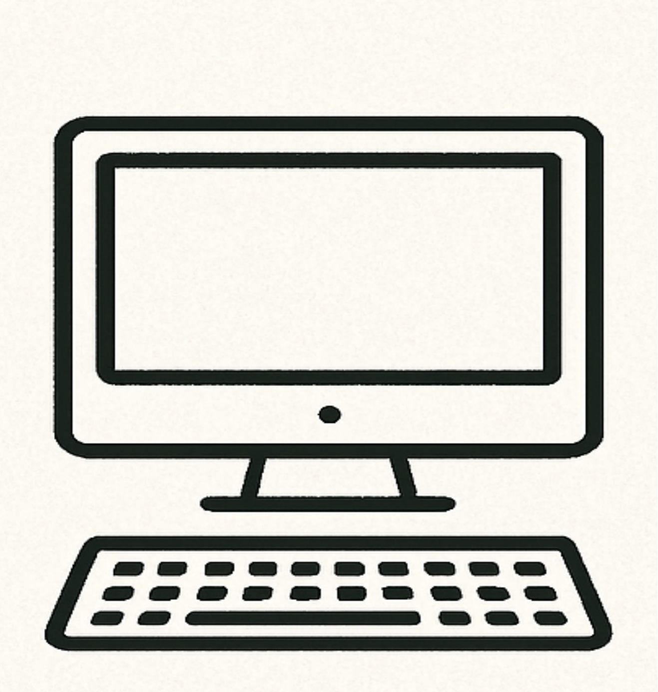
$$\mathbb{E}[Gains\ from\ trade] = 0.$$

No trade mechanism $\mathscr{M}$



Economists: Optimality is unattainable

(Informal) **Theorem [Myerson-Satterthwaite '83]:** There exists no mechanism that simultaneously guarantees individual rationality, incentive compatibility, budget balance, and maximizes Social Welfare.



Computer Scientists: Approximation Thrives

Independent Values:

- Welfare:
  - Multiple works with posted price mechanisms [Blumrosen and Dobzinski, 2014, 2021, **Cai and Wu, 2023**, Colini-Baldeschi et al., 2016, **Kang et al., 2022**, **Liu et al., 2023**].
  - State of the Art (**blue**) is a **1.38** approximation, Lower bound (**red**) is **1.354**.
- Gains From Trade:
  - Again multiple works with posted price mechanisms [**Babaioff et al., 2021**, 2020, Blumrosen and Dobzinski, 2014, Brustle et al., 2017, Cai et al., 2021, Deng et al., 2022, **Fei, 2022**, McAfee, 2008]
  - State of the Art (**blue**) achieves a **3.15** approximation, Lower bound (**red**) is **1.358**.

Correlated Values - Welfare:

- Single work [**Dobzinski and Shaulker, 2024**], that proves that a posted price mechanism achieves a **tight** **1.582** approximation.

A realistic scenario

An art connoisseur is considering selling their marble sculpture to a civil engineer.

Trade ?

Knows Signal s:
The "artistic" value of the sculpture.

Knows Signal b:
The "material" value of the sculpture.

What if they both care about the other agent's information?

Interdependent Values in Bilateral Trade

The seller has a **private** signal s and the buyer has a **private** signal b. The signals are drawn from **publicly** known distributions.

Their values for the item are **public functions** of the signals, that is the seller's value is $v_s(b\ ,\ s)$ and the buyer's value is $v_b(b\ ,\ s)$.

Motivating Interdependent Values

This is an established model, formalized by **Milgrom & Weber** in 1982. This model:

1. Describes **partial information** and **information asymmetry**,
2. Generalizes the **independent** and **correlated** values,
3. Captures unaccounted for **realistic** scenarios.

Examples:

- Resale values, Values with revision effects [Myerson '81].
- Common values - Mineral Rights model [Wilson 1969], Wallet game [Klemperer 1998].

Mechanism Constraints (Seller)

- The buyer reports their true signal b.
- The seller reports signal s' while their true signal is s.
- The seller utility is $U_S(s', s, b) = p(b, s') - v_s(s, b) \cdot x(b, s')$.

Bayesian Incentive Compatibility (BIC): A mechanism $\mathcal{M} = (x, p)$ satisfies bayesian incentive-compatibility for the seller if for every (s, s') in the seller domain:

$$\mathbb{E}_b \left[U_S(s, s, b) \right] \geq \mathbb{E}_b \left[U_S(s', s, b) \right]$$

Interim Individual Rationality (Interim IR): A mechanism $\mathcal{M} = (x, p)$ satisfies interim individual-rationality for the seller if for every s in the seller domain:

$$\mathbb{E}_b \left[U_S(s, s, b) \right] \geq 0$$

Posted Price Mechanism & Interdependent Values

- In a posted price mechanism, a take it or leave it price p is presented to both the buyer and the seller. The trade occurs **only** if they **both agree** to the price.
- Under **independent/correlated** values posted price mechanisms are trivially DSIC and ex-post IR, and always result in trade if $v_s < p < v_b$.
- Under **interdependent** values these properties are **no longer guaranteed**.

$v_s = 1$

Price $p = 1.5$?

$v_b = 2$

Posted Price Mechanism & Interdependent Values

- **Focus on the buyer**. Suppose $b = 0.5$ and a posted price $p = 1.5$;
  - The buyer calculates their value as $v_b(s) = 2 \cdot s + 0.5$;
  - How should the buyer respond to the price of 1.5?
  - The buyer should evaluate their expected value **conditioning** on what it means for the seller to also accept the trade!

$v_s = 2 \cdot b + s$

$s \sim U[0,1]$

Price $p = 1.5$?

$v_b = 2 \cdot s + b$

$b \sim U[0,1]$

(α, β)-Information Structures

- *(For ease of presentation)* Assume signals b, s drawn **independently** from $U[0,1]$ and valuations functions that are **additively separable over** the signals:

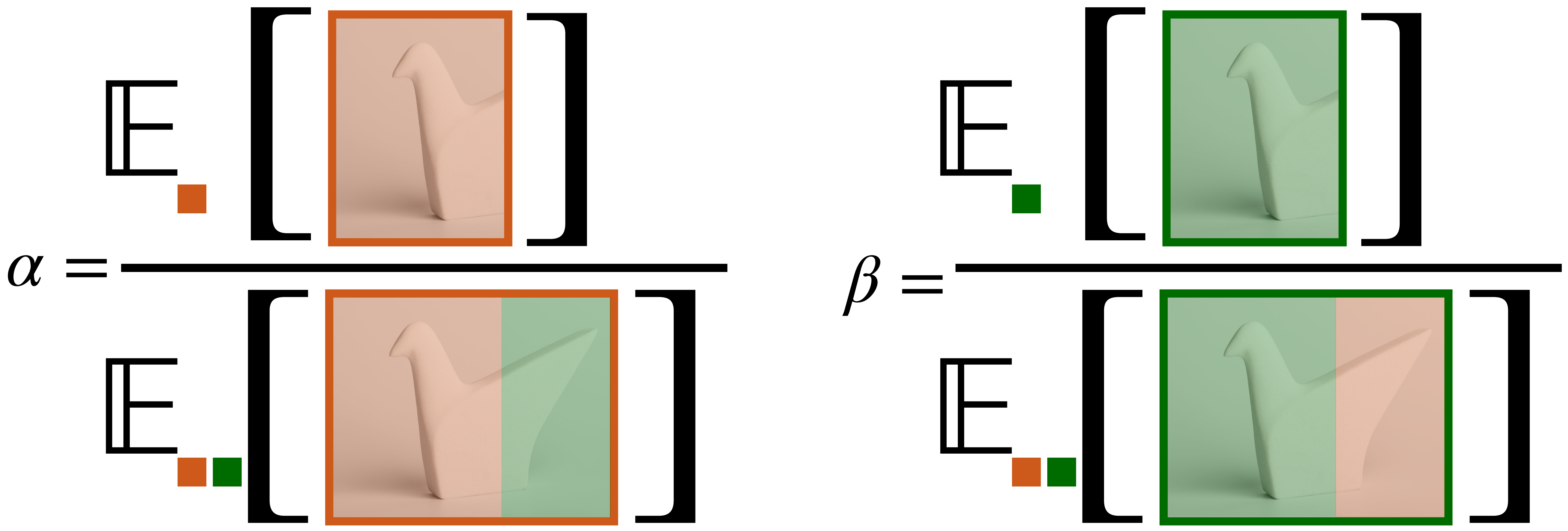
$$v_s(b, s) = f_s(b) + g_s(s), \qquad v_b(b, s) = f_b(b) + g_b(s)\,,$$

where, $f(\cdot)$, $g(\cdot)$ are non-negative, increasing functions.

- We define parameters α (seller) and β (buyer) that quantify the **influence** that an agent's **private signal has on their own valuation.** We provide results for (α, β)-information structures.

Seller and Buyer Informedness

- **Defining (α,β) pictorially:**

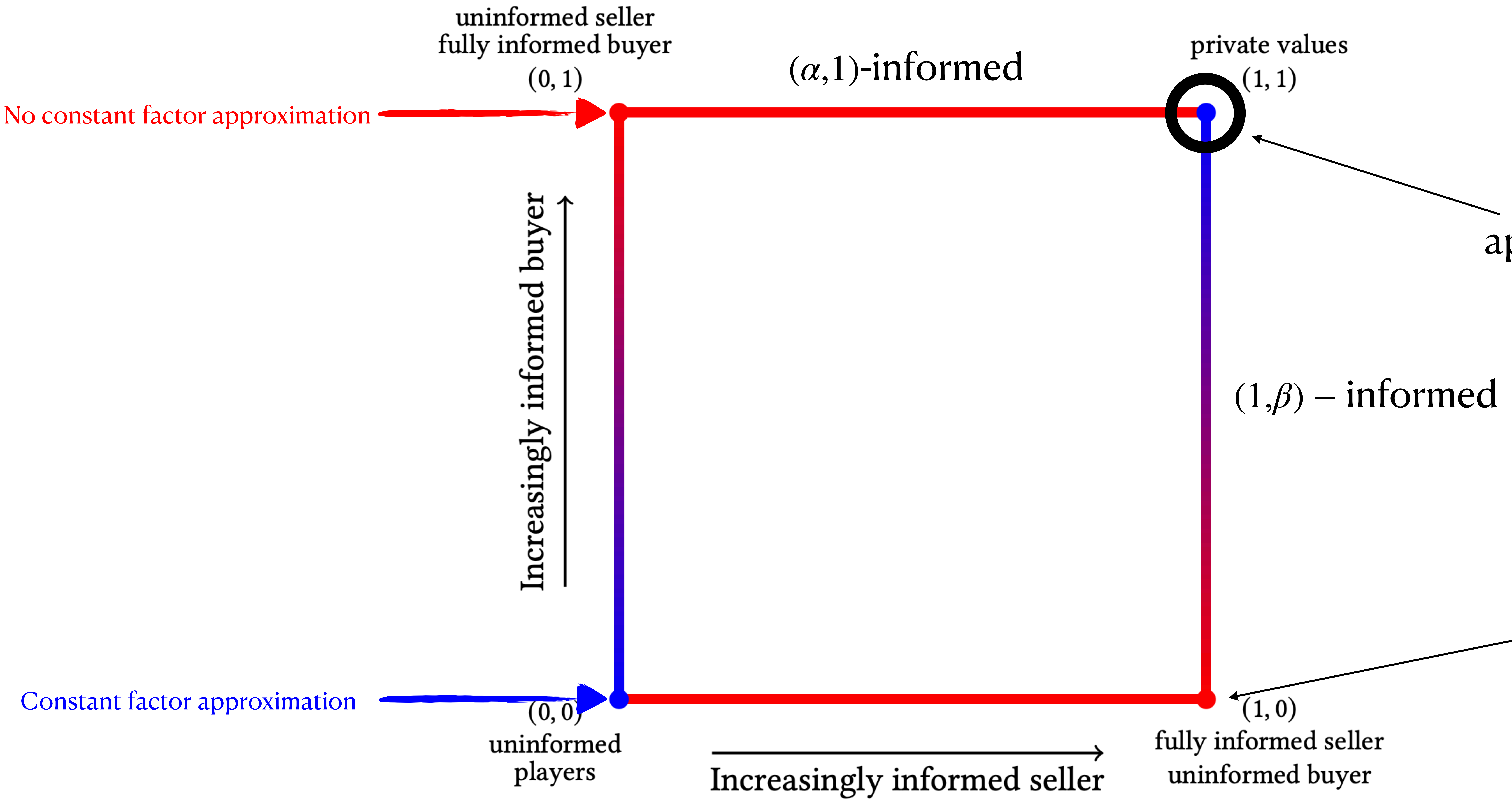


- **Uninformed** seller corresponds to $\alpha = 0$. **Fully informed** seller corresponds to $\alpha = 1$.

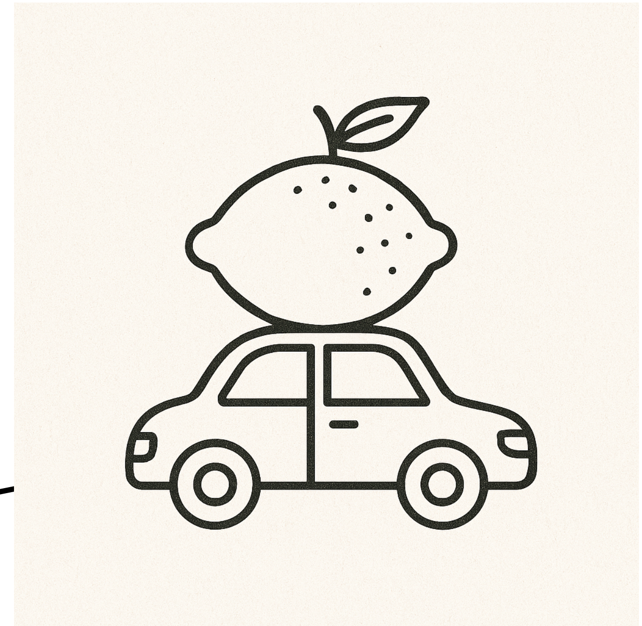
- Formally, we denote the seller α**-informed** and the buyer β**-informed** with:

$$\alpha = \frac{\mathbb{E}_s[v_s(0,s)]}{\mathbb{E}_{s,b}[v_s(b,s)]}, \qquad \beta = \frac{\mathbb{E}_b[v_b(b,0)]}{\mathbb{E}_{b,s}[v_b(b,s)]}.$$

Overview of results for (α,β) - information structures on the square

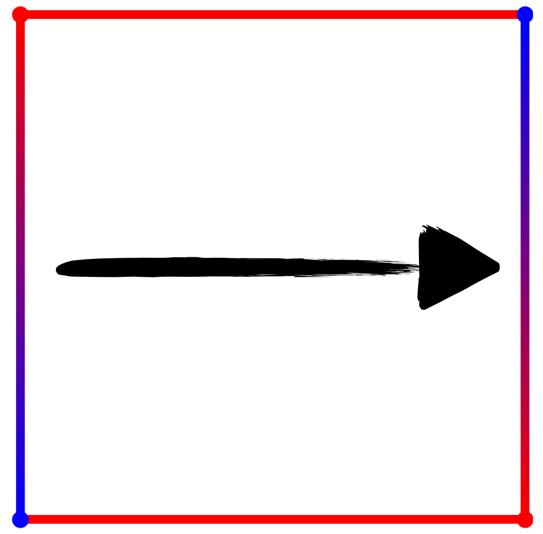


All previous
approximation result

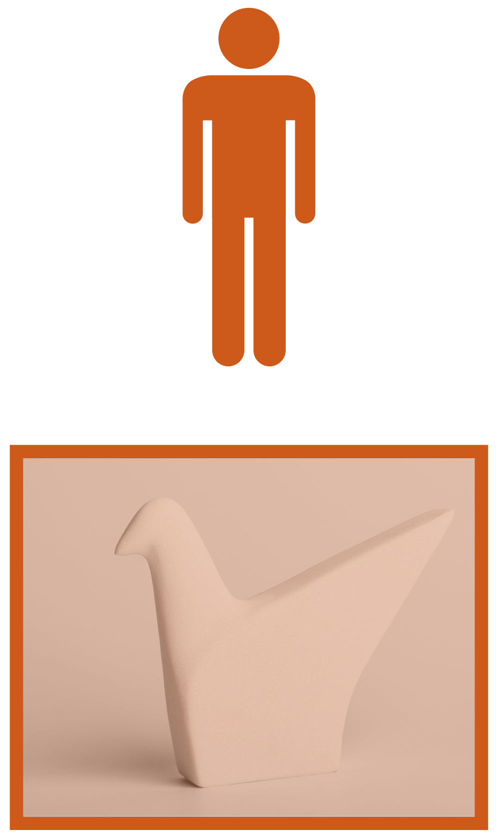


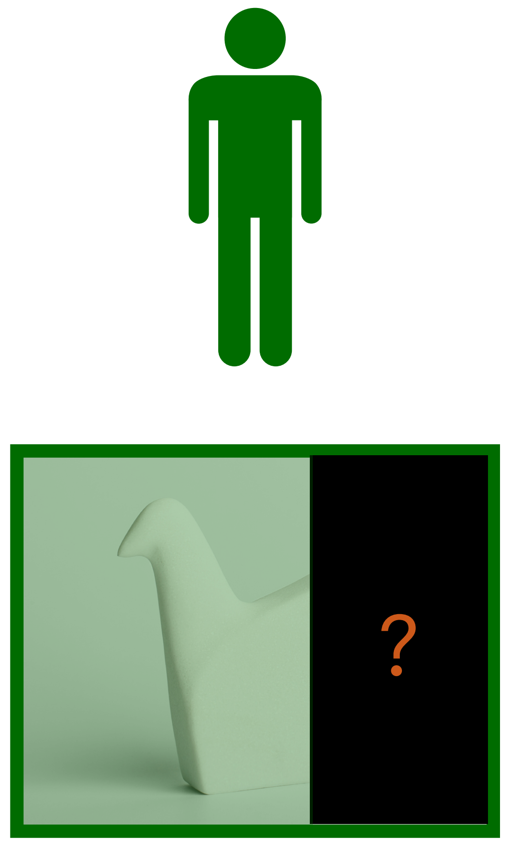
Akerlof [70]

A Fully Informed Seller: $(1, \beta)$ Edge

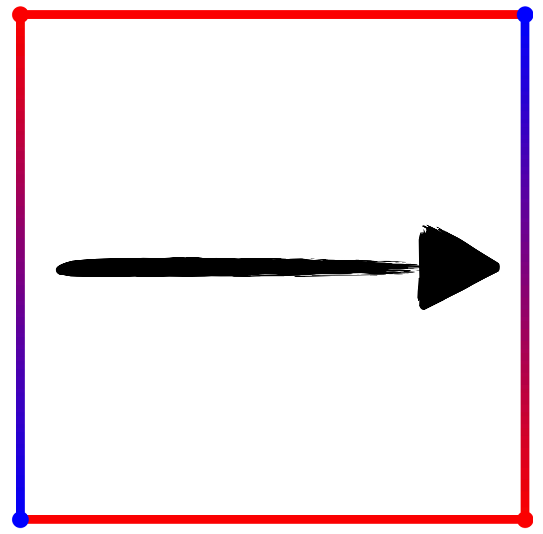


(Informal) Theorem 1: Let $\mathscr{M}$ be a posted price mechanism for the (private) independent values case with an approximation ratio of $\gamma > 1$. Consider an information structure with $\beta > 0$ and a fully informed seller ($\alpha = 1$). Then there **exists** a BIC **mechanism** $\mathscr{M}'$ with an **approximation** ratio of $\frac{2\gamma}{\beta}$.

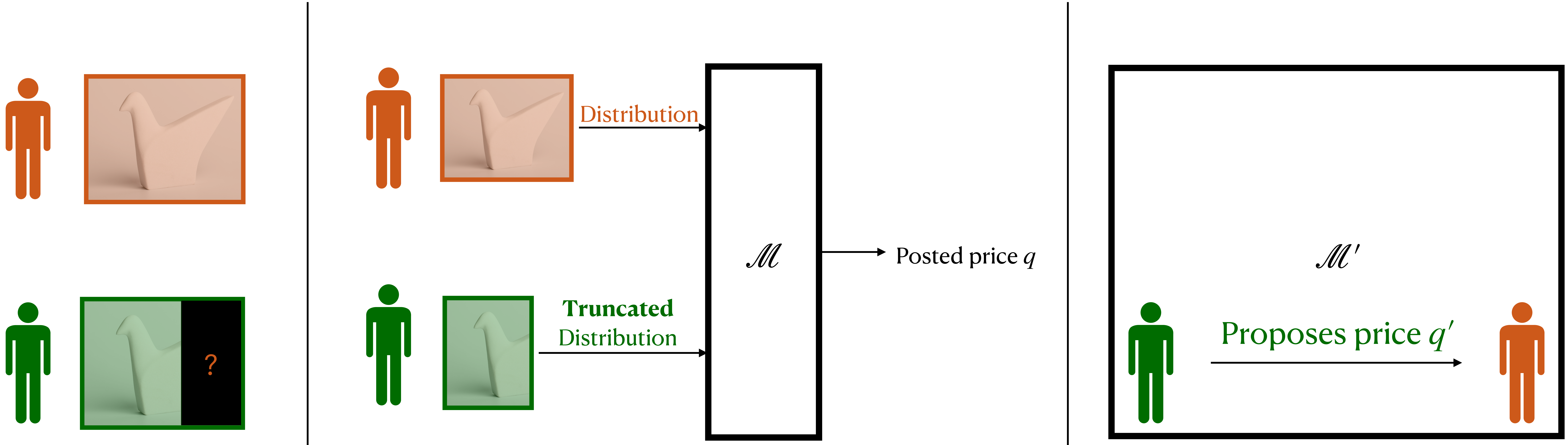




A Fully Informed Seller: (1, β) Edge

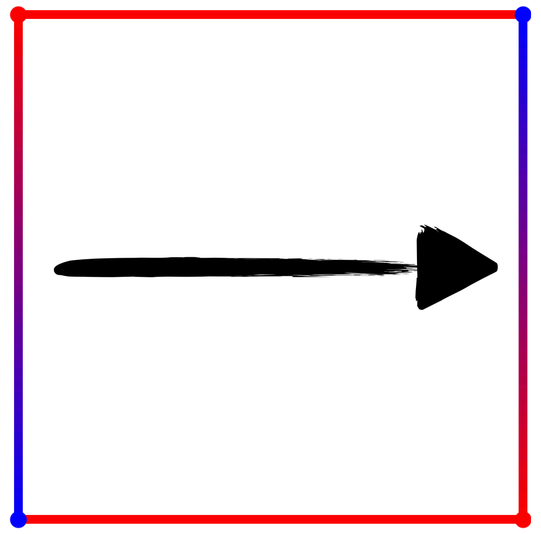


Proof Sketch - Compare Two Posted Price Mechanisms:



True Instance | **Independent** Values Instance/Mechanism | **Interdependent** Mechanism

A Fully Informed Seller: (1, β) Edge



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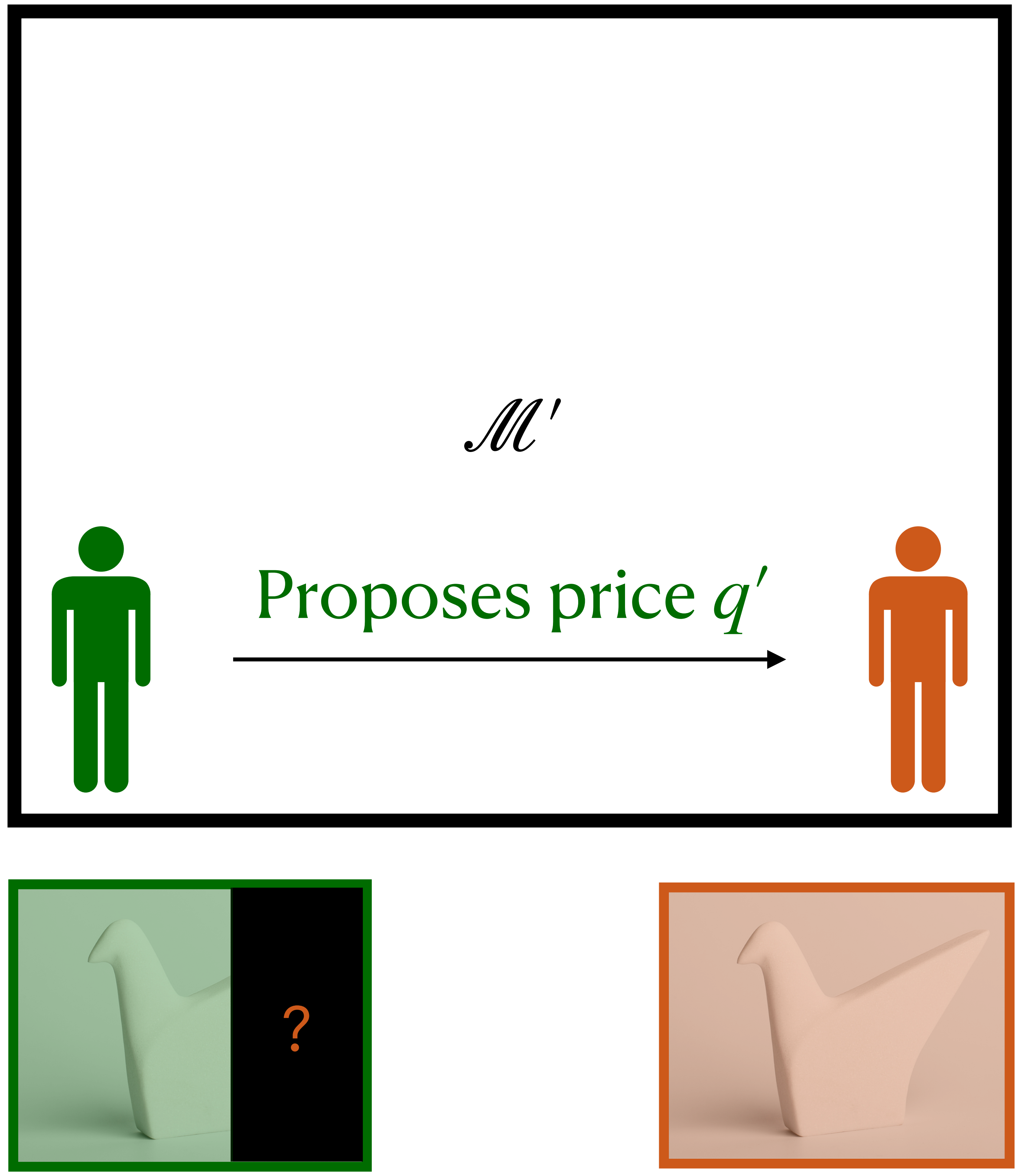
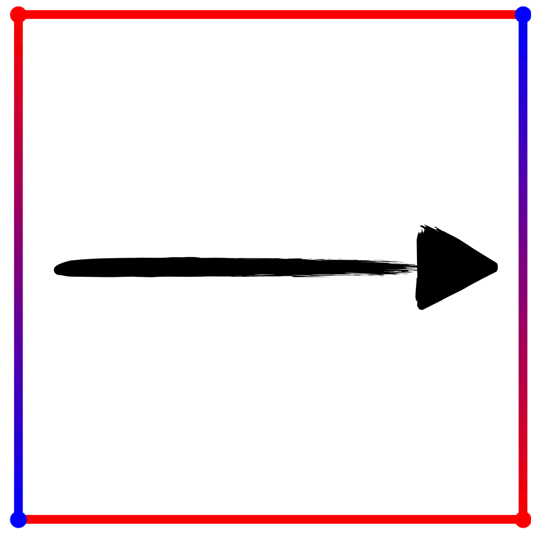


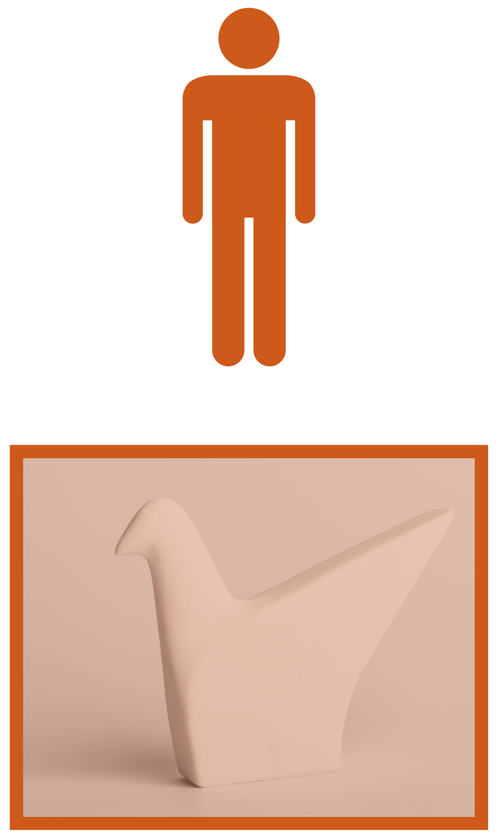
Investigating $\mathscr{M}'$:

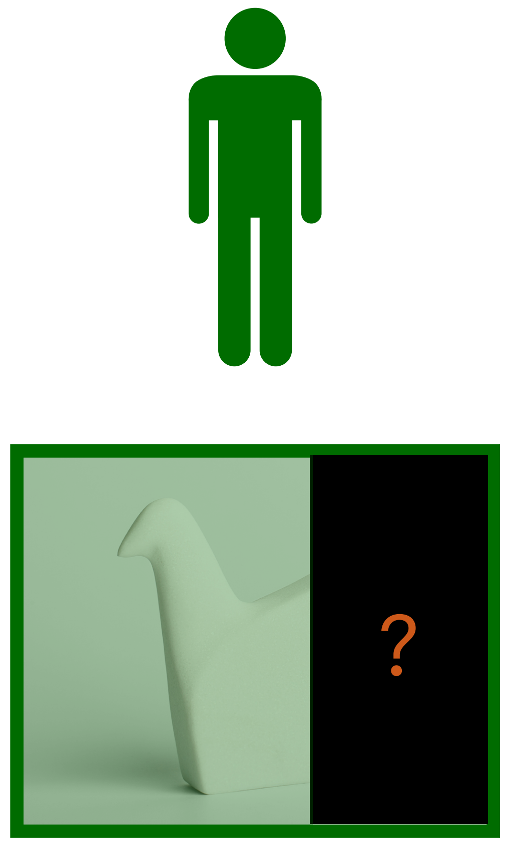
- The buyer proposes the price q' so $\mathscr{M}'$ satisfies **BIC** and **interim IR** (for the buyer).
- The seller is fully informed and responds to the proposed price optimally (so seller **BIC** and **interim IR** are also guaranteed).
- The **proposed price** q' can only be **higher** than price q (the price posted by the independent values mechanism $\mathscr{M}$).
- This implies the Welfare of $\mathscr{M}'$ is at **least as large** as the Welfare of $\mathscr{M}$.

A Fully Informed Seller: $(1, \beta)$ Edge



(Formal) Theorem 2: For every $\beta \in (0,1)$, there exists an information structure where the seller is fully informed and the buyer is β-informed, and **no** BIC and interim IR **mechanism can provide an approximation ratio** better than $\frac{2}{3\beta}$.



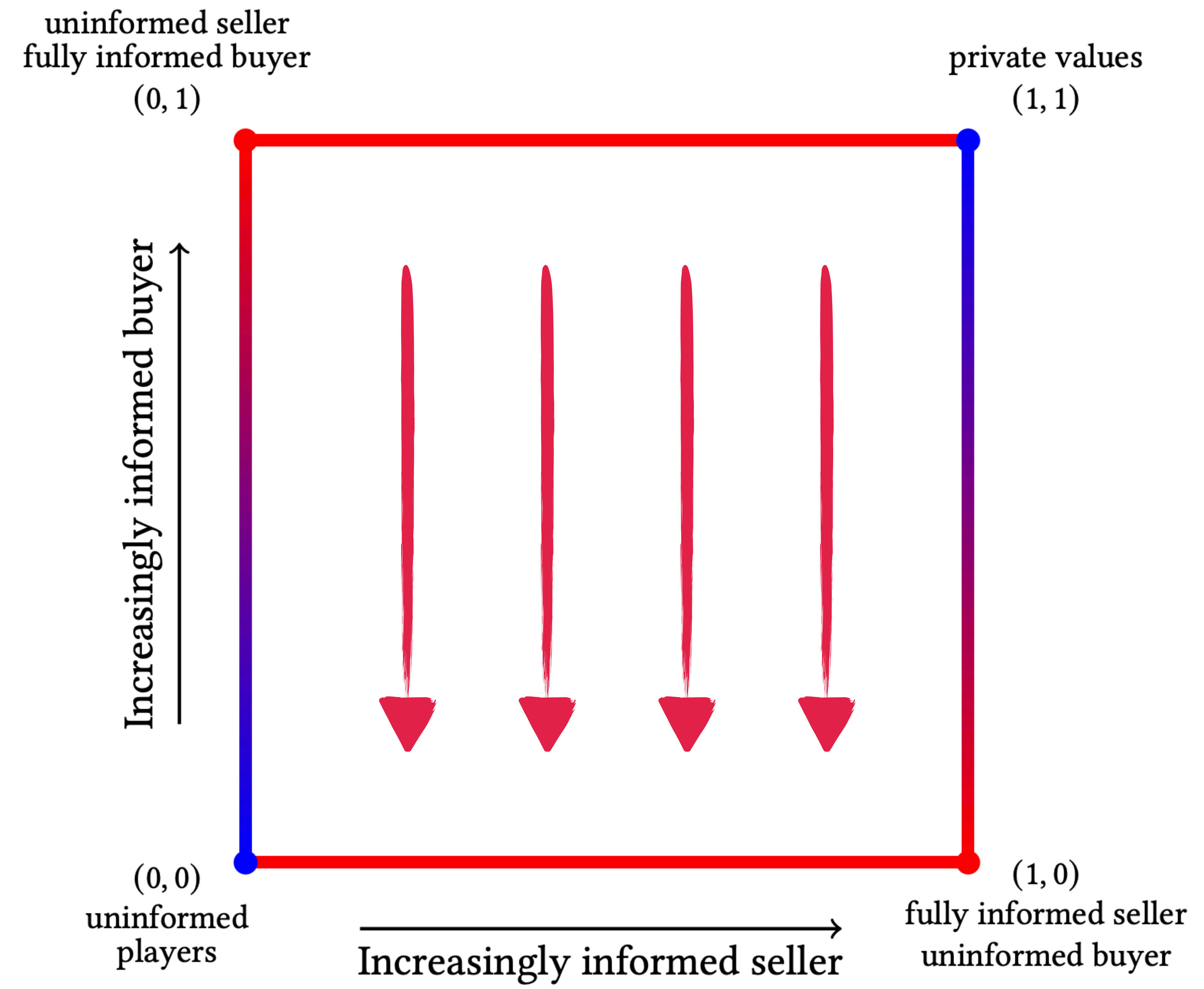


Positive result was:

$$\frac{2\gamma}{\beta}$$

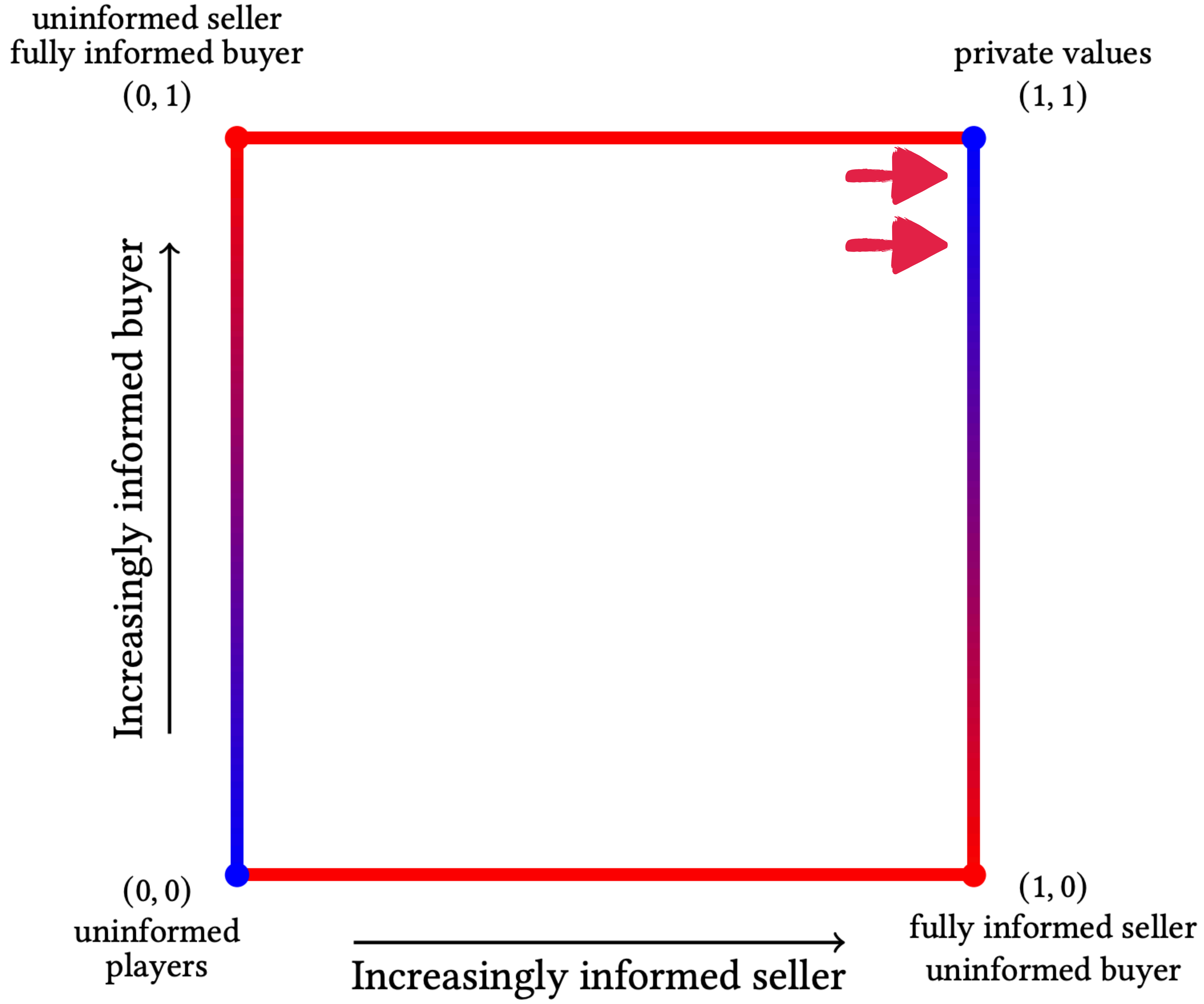
Interior of the square, overview of results

(Formal) Proposition 3: For every $\alpha > 0$ and $\beta < 1$, there exists an (α, β)-information structure where no BIC and interim IR mechanism can provide an approximation ratio better than $\frac{1}{2\beta}$.



Interior of the square, overview of results

(Formal) Proposition 4: For every $\alpha \in (0.9,1)$ and $\beta \in [1-(1-\alpha)^3,1)$, there exists an (α,β)-information structure where no BIC and interim IR mechanism can provide an approximation ratio better than $\dfrac{0.15}{1-\alpha}$.



Information Structures - Polynomials

- Assume that the valuations of the buyer and the seller are **polynomials** of the signals, of **maximum degree** k:

$$v_s(b, s) = \sum_{i=1}^{k} c_i \cdot s^i + \sum_{i=1}^{k} d_i \cdot b^i + c_s \,,$$

and

$$v_b(b, s) = \sum_{i=1}^{k} a_i \cdot b^i + \sum_{i=1}^{k} b_i \cdot s^i + c_b \,.$$

- The signals b, s are **independently** drawn from $U[0,1]$.

Polynomials - Results

(**Formal**) **Theorem 9:** Suppose that v_b, v_s are polynomials of maximum degree k, and that the signals are independently drawn from a uniform distribution over $[0,1]$. Then, there **exists** a BIC and interim IR **mechanism that guarantees an approximation** ratio of $O(k^2)$. In particular, when v_b, v_s are linear functions, the approximation ratio is constant.

(**Formal**) **Theorem 9:** For every $k \in \mathbb{N}$, there exist polynomials v_b, v_s of degree k such that **no** BIC and interim IR mechanism **can achieve an approximation ratio better** than k.

Polynomials - Approximate mechanism

Mechanism $\mathscr{M}$:

1. If $\mathbb{E}[v_s] \geq \dfrac{\mathbb{E}[v_b]}{(k+1)^2}$: Do not trade the item.

2. If $\mathbb{E}[v_s] < \dfrac{\mathbb{E}[v_b]}{(k+1)^2}$: Post a price of $q = \dfrac{\mathbb{E}[v_b]}{k+1}$

Incentives: On case 2. the seller **always agrees** to the price, while the buyer might **sometimes agree**.

Future Directions

1. **Tightly** characterize what is possible in the interior of the **square**.
2. Study other **families** of information structures.
3. Investigate what is possible for the **GFT** objective.
4. Move beyond bilateral trade to **two-sided markets** (multiple buyers and/or sellers).

Thank you !