Improved Truthful Mechanisms for Subadditive Combinatorial Auctions: Breaking the Logarithmic Barrier

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Combinatorial Auctions – The problem

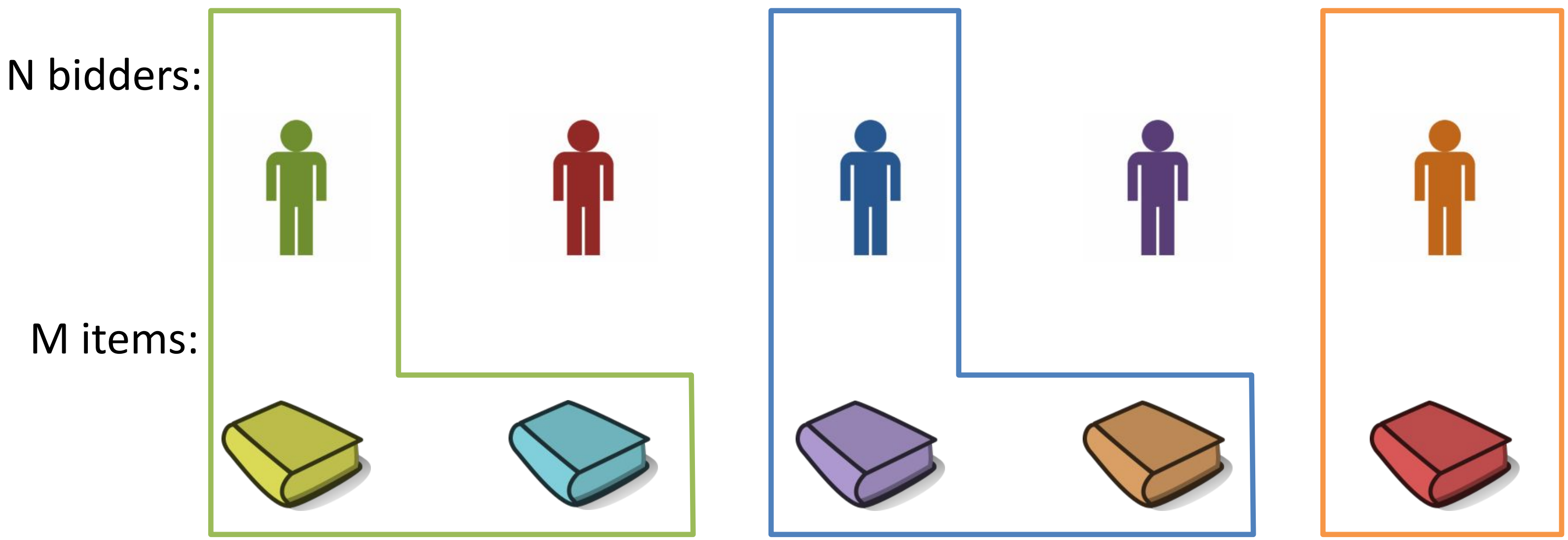
N bidders:

M items:

The objective

- The objective is to calculate an **allocation** of the items $A = (A_1, \dots, A_n)$ that maximizes the **social welfare**.
- Players have **valuation functions** v_i over the possible subsets of M.

Social Welfare = $\sum_{i \in N} v_i(A_i)$



In this case the allocation is $A = (\{1,2\}, \emptyset, \{3,4\}, \emptyset, \{5\})$

Valuations – Bidder 1

N bidders:

M items:

v {1} = 2   v {2} = 3   v {3} = 5   v {4} = 2   v {5} = 2

Valuations – Bidder 1

N bidders:

M items:

v {1} = 2 v {2} = 3 v {3} = 5 v {4} = 2 v {5} = 2

v {1,2} = 5

Valuations – Bidder 1

N bidders:

M items:

v {1} = 2   v {2} = 3   v {3} = 5   v {4} = 2   v {5} = 2

v {1,2} = 5   **v {2,3} = 8**

Valuations – Bidder 1

N bidders:

M items:

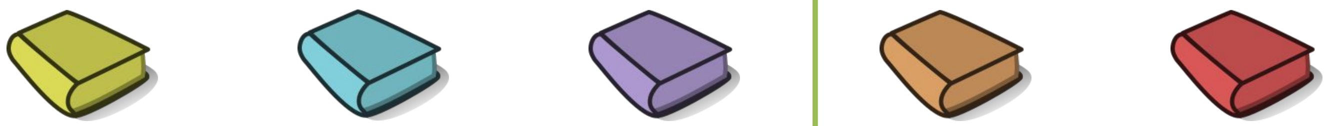
v {1} = 2 v {2} = 3 v {3} = 5 v {4} = 2 v {5} = 2

v {1,2} = 5 v {2,3} = 8 **v {3,4} = 6**

Valuations – Bidder 1

N bidders:

M items:

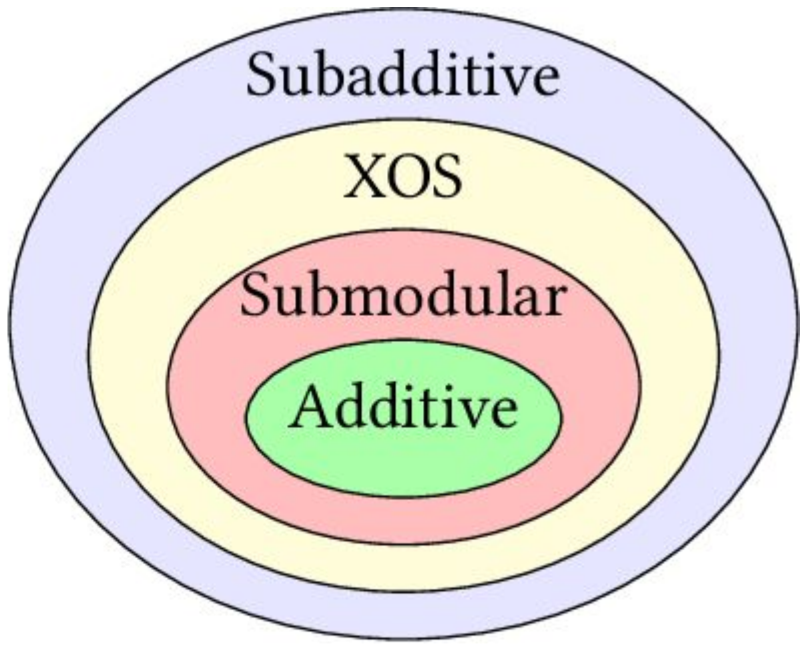


v {1} = 2 v {2} = 3 v {3} = 5 v {4} = 2 v {5} = 2

v {1,2} = 5 v {2,3} = 8 v {3,4} = 6 **v {4,5} = 3**

Valuation function classes

- **Additive:** $v(S) + v(T) = v(S \cup T) + v(S \cap T)\ \forall (S, T)$
- **Submodular:** $v(S) + v(T) \geq v(S \cup T) + v(S \cap T)\ \forall (S, T)$
- **Subadditive:** $v(S) + v(T) \geq v(S \cup T)\ \forall (S, T)$



Accessing valuation functions

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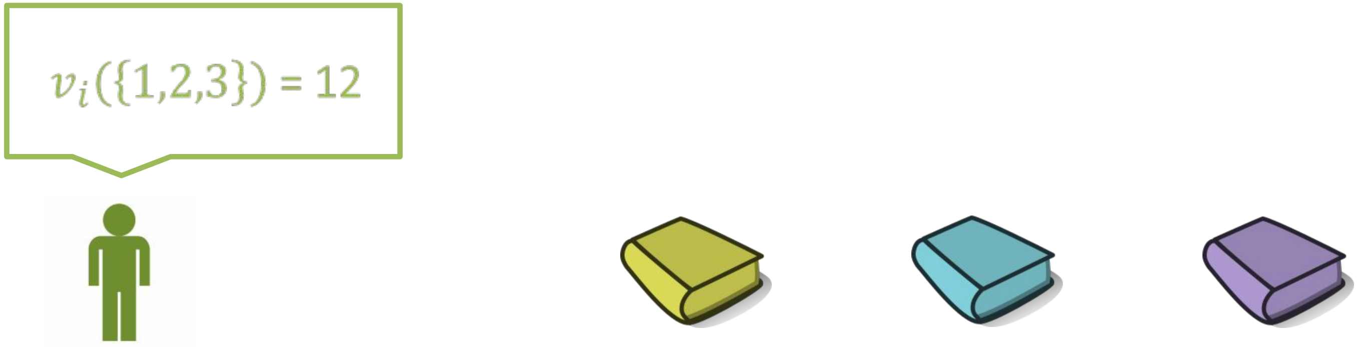
- **Valuation functions** are functions v_i defined from 2^M to R^+ .
- That means the input is exponentially large.
- **Value queries**: Presented a bundle S bidder i outputs his valuation of the bundle $v_i(S)$
- **Demand queries**: Presented a price vector **p** bidder i outputs the bundle S' that maximizes his utility, that is:

$$S' \in \arg\max_{S \subseteq M} \{v_i(S) - p(S)\}$$

Value queries:



Demand queries:

p {1} = 9 p {2} = 3 p {3} = 5

Utility – Revenue and Social Welfare

Given a price vector **p** and an allocation $A = (A_1, \dots, A_n)$ then we define:

- $Utility(i) = v_i(A_i) - p(A_i)$
- $Rev(A_i) = \sum_{e \in A_i} p(e) = p(A_i)$
- **Social Welfare** = $\sum_{i \in N} v_i(A_i) = \sum_{i \in N} (Utility(i) + Rev(A_i))$

Algorithmic Viewpoint

- Can we optimally solve the problem knowing that bidders do not misreport?
- The problem is **NP-hard** (set packing).
- There are constant approximation algorithms for the algorithmic problem for many valuation function classes.

Papers\Valuations	Submodular Approximation ratio	Subadditive Approximation ratio
B. Lehmann et al., 2006	2	
Jan Vondrák, 2006		
Dobzinski et al., 2005		
Feige and Vondrák 2006		2

Game theory - Truthfulness

- **Truthfulness** is the property that for every bidder $i \in N$ revealing their true valuation in response to certain queries is a **dominant** strategy.
- The previous algorithms **do not** capture truthfulness.

	Papers\Valuations	Submodular Approximation ratio	Subadditive Approximation ratio
	Dobzinski et al., 2006		
	P. Krysta and B. Vöcking, 2012		
	Dobzinski, 2016		
	S. Assadi, S. Singla, 2019		
	Dobzinski et al., 2005		
	Dobzinski, 2007		
This paper	**S. Assadi, T. Kesselheim, S. Singla, 2021**		

Price vectors – Learning procedures

- **Demand queries** require a price vector ***p.***
- **Truthfulness** requires that no bidder can **affect** her utility **by misreporting** her valuation.
- We need to <u>learn</u> from bidders better prices ***p*** but <u>discourage</u> bidders from taking advantage. How?
- We design mechanisms that work the problem **online**.
- **Bidders** come in **once**, answer their query, get allocated something (sometimes) and **leave**. This way we ensure learning and also truthfulness.
- What can we do with **good estimates** of the prices?

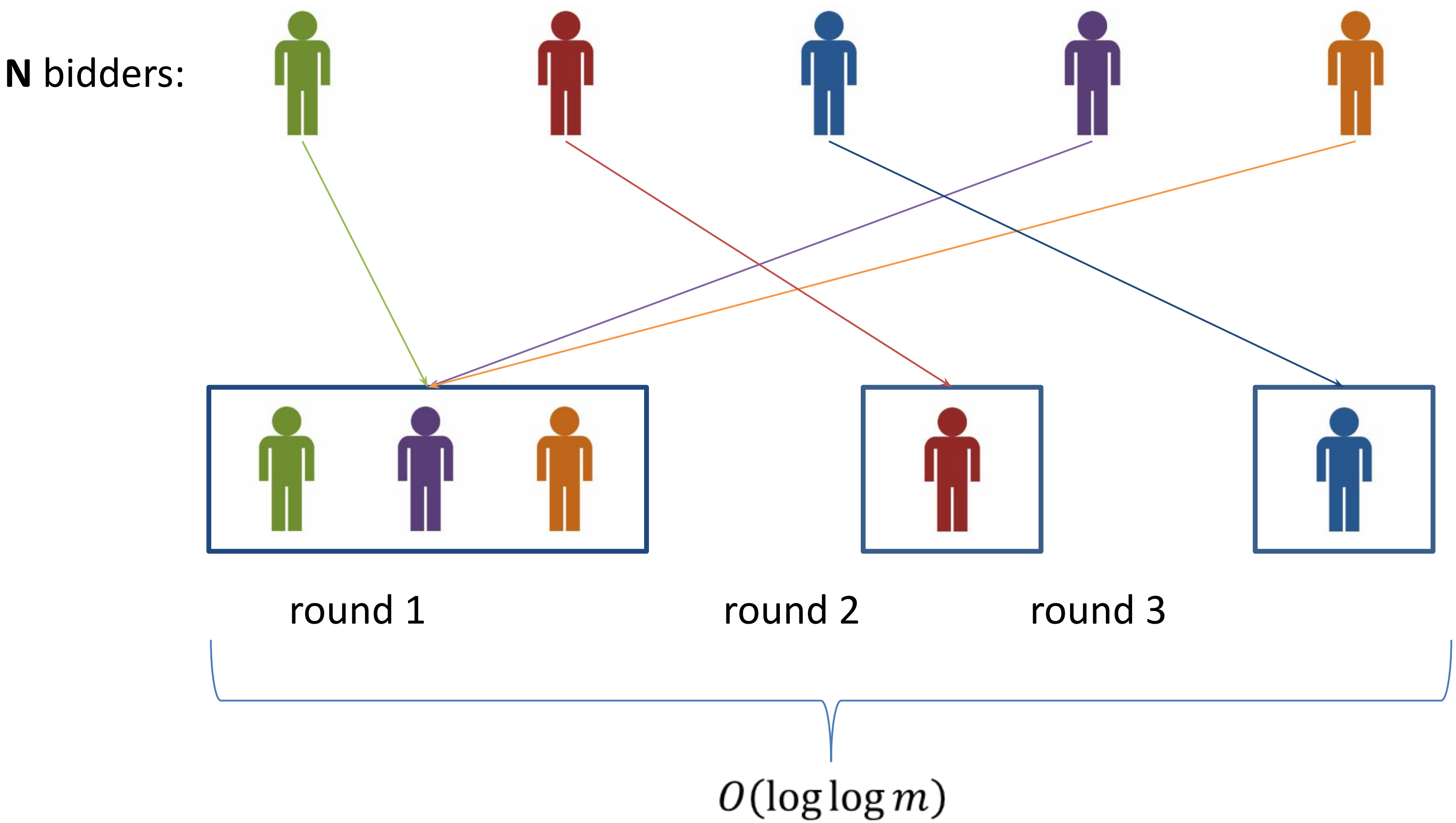
Fixed-Price Auction

- Fixed-price Auction is a format that is used as a subroutine in the mechanism. It requires a set of **bidders N**, a set of **items M**, and a **price vector p**.

FixedPriceAuction(N, M, p):

1. Iterate over the bidders i of the ordered set N in the given order:
   a) Allocate $A_i \in \arg\max_{S \subseteq M}\{vi(S) - p(S)\}$ to bidder i and update $M \leftarrow M \setminus A_i$.
2. Return the allocation $A = (A_1, \dots, A_n)$.

Randomly place bidders into rounds



Decide **randomly** that **round 2** is the winning round.

Price vectors – Round 1

Suppose for every item the **candidate price vector** is $\boldsymbol{b}_e^{(1)} = B = [0,1,2,4,\mathbf{8},16,32,64]$

Then price $p_e^{(1)} = 8$ for every item for round 1.

$O(\log m)$

M items:

$p_1^{(1)} = 8$ $p_2^{(1)} = 8$ $p_3^{(1)} = 8$ $p_4^{(1)} = 8$ $p_5^{(1)} = 8$

Round 1:

Answers Demand query with {1,3}

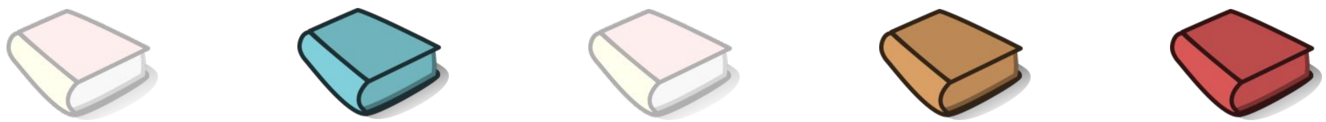
Round 1 – FixedPrice Auction

Suppose for every item the **candidate price vector** is $\boldsymbol{b}_e{}^{(1)} = B = [0,1,2,4,\mathbf{8},16,32,64]$

Then price $p_e{}^{(1)} = 8$ for every item for round 1.

$p_1{}^{(1)}=8$ $p_2{}^{(1)}=8$ $p_3{}^{(1)}=8$ $p_4{}^{(1)}=8$ $p_5{}^{(1)}=8$

M items:



$p_1{}^{(1)} = 8$ $p_2{}^{(1)} = 8$ $p_3{}^{(1)} = 8$ $p_4{}^{(1)} = 8$ $p_5{}^{(1)} = 8$

Round 1:

Answers Demand query with {2}

Round 1 – FixedPrice Auction

Suppose for every item the **candidate price vector** is $\boldsymbol{b}_e{}^{(1)} = B = [0,1,2,4,\mathbf{8},16,32,64]$

Then price $p_e{}^{(1)} = 8$ for every item for round 1.

$O(\log m)$

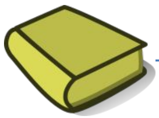
M items:

$p_1{}^{(1)} = 8$ $p_2{}^{(1)} = 8$ $p_3{}^{(1)} = 8$ $p_4{}^{(1)} = 8$ $p_5{}^{(1)} = 8$

Round 1:

Answers Demand query with {Ø}

Round 1 finishes



Sold

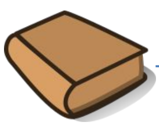
$\boldsymbol{b}_1^{(2)} =$ [~~0,1,2,4~~,8,16,32,64] , $p_1^{(2)} = 32$

Sold

$\boldsymbol{b}_2^{(2)} =$ [~~0,1,2,4~~,8,16,32,64], $p_2^{(2)} = 32$

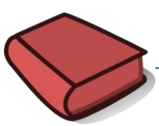
Sold

$\boldsymbol{b}_3^{(2)} =$ [~~0,1,2,4~~,8,16,32,64], $p_3^{(2)} = 32$



Not Sold

$\boldsymbol{b}_4^{(2)} =$ [0,1,2,4,~~8,16,32,64~~], $p_4^{(2)} = 2$



Not Sold

$\boldsymbol{b}_5^{(2)} =$ [0,1,2,4,~~8,16,32,64~~], $p_5^{(2)} = 2$

Round 2 begins (winning round)

$p_1{}^{(2)} = 32$ $p_2{}^{(2)} = 32$ $p_3{}^{(2)} = 32$ $p_4{}^{(2)} = 2$ $p_5{}^{(2)} = 2$

Round 2:

Answers Demand query with {2,4,5}

- No other players in this round. The auction **completes** since this is the winning round, and allocates items {2,4,5} to the red bidder.
- Notice that green, orange and purple participated in the auction but got nothing and also that blue didn't get a chance to participate.

The Binary-Search Mechanism

- Split bidders (uniformly at random) into $\beta + 1 = \log|B| = O(\log\log m)$ groups - rounds that we will be handling sequentially.
- Select uniformly at random a round $r^\star$, and consider it the winning round.
- Define a price vector B of **candidate** prices. The size of this vector B is $O(\log m)$. Define for every item e, $\boldsymbol{b}_e{}^{(1)} = B$
- For $\ell = 1$ to β rounds:

  1. For every item e, define the price $p_e{}^{(\ell)} = b_{e,\frac{k}{2}+1}{}^{(\ell)}$, where $k = {}^{|B|}/_{2^{\ell-1}}$ and $\boldsymbol{b}_e{}^{(\ell)} = \left(b_{e,1}{}^{(\ell)}, \dots, b_{e,k}{}^{(\ell)}\right)$ is the vector of candidate prices for e.
  2. Run Fixed-Price Auction(N_ℓ, M, $\boldsymbol{p}^{(\ell)}$), where N_ℓ is the set of bidders participating in this round, and let $A(\ell)$ be the allocation.
  3. For every item e, if e is allocated in $A(\ell)$, define remaining candidate prices $\boldsymbol{b}_e{}^{(\ell+1)} = \left(b_{e,\frac{k}{2}+1}{}^{(\ell)}, \dots, b_{e,k}{}^{(\ell)}\right)$, and otherwise define $\boldsymbol{b}_e{}^{(\ell+1)} = \left(b_{e,1}{}^{(\ell)}, \dots, b_{e,\frac{k}{2}}{}^{(\ell)}\right)$.
  4. If $\ell = r^\star$ then return $A^{(\ell)}$ as the final allocation.

- Run Fixed-Price Auction($\mathrm{N}_{\beta+1}$,M,$\boldsymbol{p}^{(\beta+1)}$), where $p_e{}^{(\beta+1)}$ in $\boldsymbol{p}^{(\beta+1)}$ is the unique price in $\boldsymbol{b}_e{}^{(\beta+1)}$ and return this as the final allocation.

Standard assumptions and notation

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- We have knowledge of $\psi = \max_i v_i(M)$. That is essentially an upper bound for all valuations functions. This can be **removed**.
- Valuation functions are:
  - **Normalized** $v(\emptyset) = 0$
  - **Monotone** $v(S) \leq v(T)\ \forall S \subseteq T \subseteq M$
- ${p_e}^{(\ell)}$ is the price we choose for the FixedPrice Auction for item e in round ℓ.
- $\beta + 1 = \log|B|$ is the number of rounds.
- r_i denotes the round bidder **i** is placed in.
- $r^\star$ is the **winning round**.
- r_{-i} denotes a vector containing $r_{i'}$ for all $\boldsymbol{i}' \neq \boldsymbol{i}$. So it describes the rounds everyone else participates in. (i.e. [1,3,1,5,4,2,_,5,2,3])
- $\boldsymbol{q}(S) = \sum_{e \in S} q_e$.
- $S_i^\star$ will be used to describe the set that bidder i would get in the optimal allocation.

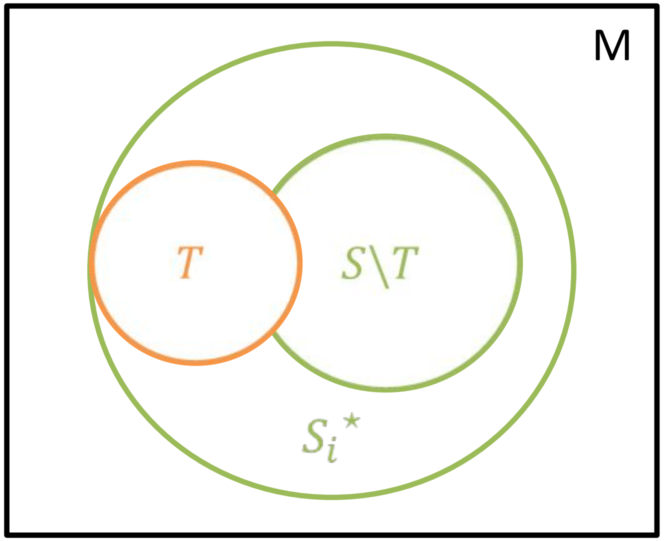
Lemma 1([DKL20])

$\psi = max_i v_i(M)$: Valuation Upper **bound**
$p_e^{(\ell)}$: item e **price** in round ℓ.
$\beta = \log|B|$: number of **rounds**.
r_i: bidder **i**'s round.
r_{-i}: everyone **but i**'s round.
$r^\star$: **Winning** round.
$S_i^\star$: **Optimal** allocation for i.

Lemma 1: Given **any set** of items $U \subseteq M$ and valuation $v: 2^M \to R^+$, there exists a **distribution** λ over item sets $S \subseteq U$ and **prices** $\boldsymbol{q}^\star \in R^{|U|} \geq 0$ such that for **any set** $T \subseteq U$, we have:

$$\underbrace{\sum_{S\subseteq U} \lambda_S \left(v(S \setminus T) - \boldsymbol{q}^\star(S)\right)}_{\text{Expected Utility}} + \underbrace{\boldsymbol{q}^\star(T)}_{\text{"Lost" Revenue}} \geq \frac{1}{\alpha} \cdot v(U).$$

- **Subadditive** valuation v then $\alpha = O(\log\log m)$
- **XOS** valuation v then $\alpha = O(1)$.



$$E[Utility(S\backslash T)] + Rev(T) \geq \frac{1}{\alpha} v(S_i^\star)$$

That holds $\forall S, T \subseteq S_i^\star$ for some distribution λ and some price vector $\boldsymbol{q}^\star$

$\psi = max_i v_i(M)$: Valuation Upper **bound**
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Binary search - Rounding prices

- **Observation 1:** For a given ψ, use $B := \{0, 2^{-3\lceil \log m \rceil}\psi, 2^{1-3\lceil \log m \rceil}\psi, \dots, 2^{-1}\psi\}$ as a set of $3\lceil \log m \rceil$ candidate prices. Now for any set of items $U \subseteq M$ and any bidder $i \in N$, if $v_i(U) \leq \psi$ then there exist a distribution λ over item sets $S \subseteq U$ and prices $\boldsymbol{q} \in B^{|U|}$ such that for any $T \subseteq U$:

$$\sum_{S \subseteq U} \lambda_S \left(v_i(S \setminus T) - \boldsymbol{q}(S)\right) \geq \frac{1}{\alpha} \cdot v_i(U) - \mathbf{2}\boldsymbol{q}(T) - \frac{\psi}{m^2}.$$

- **Subadditive** valuation v_i then $\alpha = O(\log \log m)$.
- **XOS** valuation v_i then $\alpha = O(1)$.

$\boldsymbol{\psi = max_i v_i(M)}$: Valuation Upper **bound**
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Theorem - Result

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Theorem 1: For a combinatorial auction with n subadditive bidders and m items, given $\psi \geq \max_i v_i(M)$, BinarySearchMechanism is universally **truthful**, uses at most **n demand queries** and **polynomial time**, and has **expected welfare**:

$$E[Welfare] \geq \frac{1}{2\cdot\alpha\cdot(\beta+1)^2} \sum_i v_i\left(S_i^\star\right) - \frac{\psi}{m},$$

- **Subadditive** valuation v_i then $\alpha = O(\log\log m)$.
- **XOS** valuation v_i then $\alpha = O(1)$.

Proposition

$\boldsymbol{\psi = max_i v_i(M)}$: Valuation Upper **bound**.
$p_e^{(\ell)}$: item e **price** in round ℓ.
$\beta = log|B|$: number of **rounds**.
r_i: bidder **i**'s round.
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$r^\star$: **Winning** round.
$S_i^\star$: **Optimal** allocation for i.

- **Proposition 1:** Consider any bidder $i \in N$ and the case that $\psi \geq v_i(S_i^\star)$. Let r_{-i} denote a vector containing $r_{i'}$ for all $i' \neq i$. Then, for any choices of r_{-i} we have

$$E_{r^\star,\, r_i}\left[Utility(i) + Rev\left(S_i^\star\right)|r_{-i}\right] \geq \frac{1}{2\cdot\alpha\cdot(\beta+1)^2} v_i\left(S_i^\star\right) - \frac{\psi}{m^2},$$

where $Utility$ and Rev are defined for the Fixed-Price Auction corresponding to the final allocation (winning round).

Using this we can prove the **Theorem 1:**

$$E[Welfare] = \sum_i E\left[Utility(i) + Rev\left(S_i^\star\right)\right]$$

$$\geq \sum_{i:S_i^\star \neq \emptyset} E\left[Utility(i) + Rev\left(S_i^\star\right)\right]$$

$$= \sum_{i:S_i^\star \neq \emptyset} E_{r_{-i}} E_{r^\star,\, r_i}\left[Utility(i) + Rev\left(S_i^\star\right)|r_{-i}\right] \geq \frac{1}{2\cdot\alpha\cdot(\beta+1)^2} \sum_i v_i\left(S_i^\star\right) - \frac{\psi}{m},$$

where the last inequality is because $\left|\left\{i: S_i^\star \neq \emptyset\right\}\right| \leq m$ and **Proposition 1**.

Proof of Proposition 1

$\boldsymbol{\psi} = \boldsymbol{max_i v_i(M)}$: Valuation Upper **bound**
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-

We will be viewing **each bidder independently**. We consider the execution of BinarySearchMechanism($\{\boldsymbol{N}\backslash \text{i}\}$, M, ψ). For $e \in S_i^\star$ we let q_e denote the item price given in Observation 1. That is what we will be calling the **correct price**.

There are three possible scenarios for every $e \in S_i^\star$:

1. $p_e^{(\beta+1)} = q_e$, we hit the **correct price**.
2. $p_e^{(\beta+1)} > q_e$, $\exists$ round $\ell \in [\beta]$ where $e \in A^\ell$ and also $p_e^{(\ell)} > q_e$.
3. $p_e^{(\beta+1)} < q_e$, $\exists$ round $\ell \in [\beta]$ where $e \notin A^\ell$ although $p_e^{(\ell)} \leq q_e$.

$S_i^\star$ partition

$\psi = max_i v_i(M)$: Valuation Upper **bound**
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-

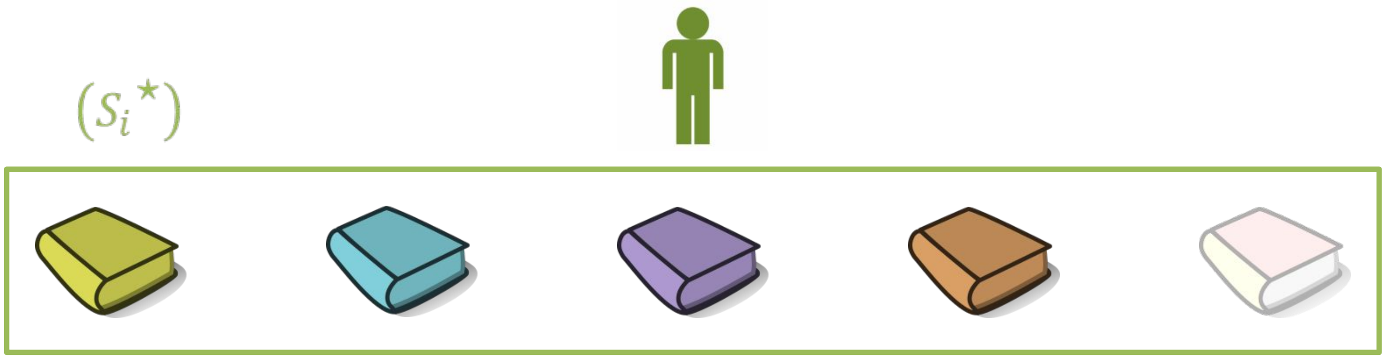
Based on these scenarios we partition $S_i^\star$ as follows:

- $C \subseteq S_i^\star$ is defined as $\{e \in S_i^\star \mid p_e^{(\beta+1)} = q_e\}$.
- $D \subseteq C$ is defined as the items sold in FixedPriceAuction before bidder i's turn.
- $O_\ell \subseteq S_i^\star$ is defined as $\{e \in S_i^\star \mid p_e^{(\ell)} > q_e \; and \; e \in A^\ell\}$. ℓ is the lowest round this occurs.
- $U_\ell \subseteq S_i^\star$ is defined as $\{e \in S_i^\star \mid p_e^{(\ell)} \leq q_e \; and \; e \notin A^\ell\}$. ℓ is the lowest round this occurs.

Note that $S_i^\star = \bigcup_{\ell=1}^{\beta}(U_\ell \cup O_\ell) \cup C$

We also define $T \equiv \bigcup_{\ell=1}^{\beta}(O_\ell) \cup D$ as the set of items which are **sold** at a **sufficiently high price**.

Tracking progress on round β+1



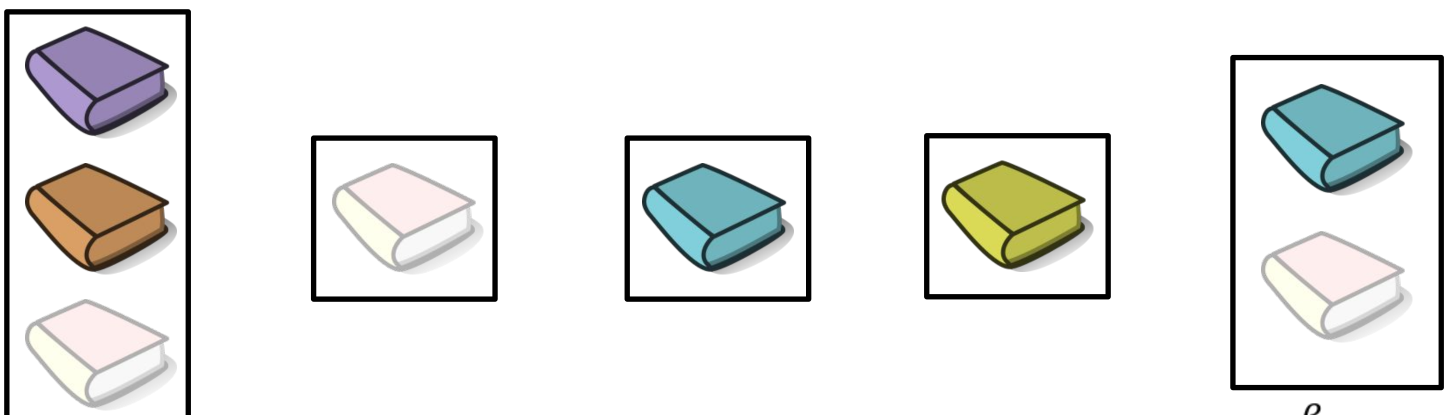
$p_1{}^{(\beta+1)} = 8$, $q_1 = 16$

$p_2{}^{(\beta+1)} = 4$, $q_2 = 2$

$p_3{}^{(\beta+1)} = 16$, $q_3 = 16$

$p_4{}^{(\beta+1)} = 4$, $q_4 = 4$

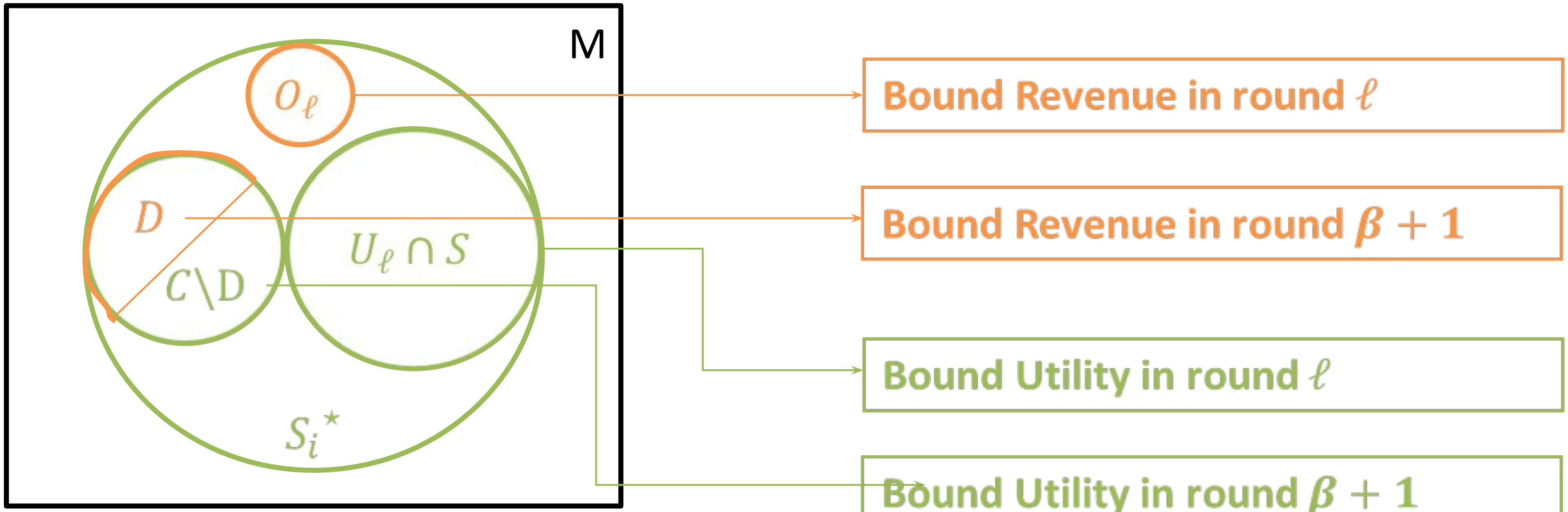
$p_5{}^{(\beta+1)} = 64$, $q_5 = 64$



$C \subseteq S_i{}^{\star}$ $D \subseteq C$ $O_\ell \subseteq S_i{}^{\star}$ $U_\ell \subseteq S_i{}^{\star}$ $T \equiv \bigcup_{\ell=1}^{\beta} (O_\ell) \cup D$

- We will use T to bound the $E[Rev(S_i^\star)]$ and $S \setminus T$ to bound the $E[Utility(i)]$.

Note that the first expectation over the revenue of the items within $S_i^\star$ (almost irrelevant who gets them) while the second expectation is over a specific bidder i.



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$S_i^\star$: **Optimal** allocation for i.

Revenue Bound

Claim 1: For any choice of r_{-i}, we have:

$$E_{r^\star, r_i}\left[Rev\left(S_i^\star\right)|r_{-i}\right] \geq \frac{1}{(\beta+1)^2} \cdot \boldsymbol{q}(T)$$

Proof:

- If $r_i \neq \beta + 1$ we use the trivial bound $Rev\left(S_i^\star\right) = 0$
- If $r_i = \beta + 1$ and $r^\star = \ell < \beta + 1$ we argue that $Rev\left(S_i^\star\right) \geq \mathbf{q}(O_\ell)$
- If $r_i = \beta + 1 = r^\star$ we argue that $Rev\left(S_i^\star\right) \geq \mathbf{q}(D)$

So for every combination of $r_i, r^\star$ $\left((\beta + 1)^2\right)$ we have:

$$E_{r^\star, r_i}\left[Rev\left(S_i^\star\right)|r_{-i}\right] \geq \frac{1}{(\beta + 1)^2}\left[\left(\sum_{\ell=1}^{\beta} \mathbf{q}(O_\ell)\right) + \mathbf{q}(D)\right] =$$

$$\frac{1}{(\beta+1)^2} \cdot \boldsymbol{q}(T) \qquad \square$$

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Utility Bound

Claim 2: For any choice of r_{-i}, and any arbitrary set $S \subseteq S_i^\star$, we have:

$$E_{r^\star,\, r_i}[Utility(i)|r_{-i}] \geq \frac{1}{(\beta+1)^2}(v_i(S \backslash T) - \boldsymbol{q}(S \backslash T))$$

Proof:

- If $r_i \neq r^\star$, trivially $Utility(i) = 0$
- If $r_i = r^\star = \ell < \beta + 1$, then $Utility(i) \geq v_i(U_\ell \cap S) - q(U_\ell \cap S)$
- If $r_i = r^\star = \beta + 1$, then $Utility(i) \geq v_i\big((C \backslash D) \cap S\big) - \boldsymbol{q}\big((C \backslash D) \cap S\big)$

Again for every combination of $r_i, r^\star$ we have:

$$E_{r^\star,\, r_i}[Utility(i)|r_{-i}]$$
$$\geq \frac{1}{(\beta+1)^2}\left\{\sum_{\ell=1}^{\beta} v_i(U_\ell \cap S) + v_i\big((C \backslash D) \cap S\big) - \boldsymbol{q}\big((C \backslash D) \cap S\big) - \sum_{\ell=1}^{\beta} \boldsymbol{q}(U_\ell \cap S)\right\}$$
$$\geq \frac{1}{(\beta+1)^2}(v_i(S \backslash T) - \boldsymbol{q}(S \backslash T))$$

Where the last inequality is because of subadditivity of v_i, and $S_i^\star = \cup_{\ell=1}^{\beta}(U_\ell \cup O_\ell) \cup C$ □

Finally bounding welfare

$\boldsymbol{\psi} = \boldsymbol{max_i v_i(M)}$: Valuation Upper **bound**
$p_e^{(\ell)}$: item e **price** in round ℓ.
$\beta = log|B|$: number of **rounds**.
r_i: bidder **i**'s round.
r_{-i}: everyone **but i**'s round.
$r^\star$: **Winning** round.
$S_i^\star$: **Optimal** allocation for i.

- Given $Utility(i) \geq 0$ we can **relax** the utility bound to:

$$E_{r^\star, r_i}[Utility(i)|r_{-i}] \geq \frac{1}{\mathbf{2}(\beta+1)^2}(v_i(S\backslash\mathrm{T}) - \boldsymbol{q}(S\backslash\mathrm{T}))$$

Summing **Claim 1** and **Claim 2** we get:

$$E_{r^\star, r_i}\left[Utility(i) + Rev\left(S_i^\star\right)|r_{-i}\right] \geq \frac{1}{2(\beta+1)^2}(v_i(S\backslash\mathrm{T}) - \boldsymbol{q}(S\backslash\mathrm{T})) + \frac{1}{(\beta+1)^2}\cdot\boldsymbol{q}(T)$$

Using **Observation 1** we get:

$$\sum_{S\subseteq S_i^\star}\lambda_S(v_i(S\backslash\mathrm{T}) - \boldsymbol{q}(S\backslash\mathrm{T})) \geq \sum_{S\subseteq S_i^\star}\lambda_S(v_i(S\backslash\mathrm{T}) - \boldsymbol{q}(S)) \geq \frac{1}{a}\cdot v_i\left(S_i^\star\right) - 2\boldsymbol{q}\ (T) - \frac{\psi}{m^2}$$

Finally, **combining these two** inequalities with the fact that $\sum_{S\subseteq S_i^\star}\lambda_S \leq 1$ we get:

$$E_{r^\star, r_i}\left[Utility(i) + Rev\left(S_i^\star\right)|r_{-i}\right] \geq \frac{1}{2(\beta+1)^2}\sum_{S\subseteq S_i^\star}\lambda_S(v_i(S\backslash\mathrm{T}) - \boldsymbol{q}(S\backslash\mathrm{T})) + \frac{1}{(\beta+1)^2}\cdot\boldsymbol{q}(T)$$

$$\geq \frac{1}{2(\beta+1)^2}\left(\frac{1}{a}\cdot v_i\left(S_i^\star\right) - 2\boldsymbol{q}\ (T) - \frac{\psi}{m^2}\right) + \frac{1}{(\beta+1)^2}\cdot\boldsymbol{q}(T) \geq \frac{1}{2\cdot\alpha\cdot(\beta+1)^2}v_i\left(S_i^\star\right) - \frac{\psi}{m^2}$$

$\square$

Summing up

We discussed:

- Valuation functions classes.
- Access Queries.
- Truthfulness.
- The Binary-Search Mechanism.
- Lemma of existence of prices and distribution.
- Proof of Binary-Search Mechanism approximation.

Final Questions?

Combinatorial Auctions

- We have n bidders (in N) and m items (in M) to be allocated.
- Every bidder $i \in N$ has a **valuation function** $v_i\colon 2^M \rightarrow R^+$ that describes the value of every bundle $S \subseteq M$ for bidder i.
- The objective is to design a mechanism that presents an **allocation** of the items $A = (A_1, \dots, A_n)$ that maximizes the **social welfare**.
- We require our mechanism to be **computationally efficient** and **truthful**.
- **Truthfulness** is the property that for every bidder $i \in N$ revealing their true valuation in response to certain queries is a **dominant** strategy.

Definitions and notations

- The mechanism devised in this paper is an **online** procedure (bidders come in 1-by-1), are presented some price vector **p** and answer one **demand queries**.
- **Demand queries**: Presented a price vector **p** bidder i outputs the bundle S that maximizes his utility, that is:

$$S \in argmax_{S' \subseteq M}\{v_i(S') - p(S')\}$$

- $\boldsymbol{Utility}(\boldsymbol{i})$ for a bundle S' given a price vector **p** is defined as:

$$Utility(i) = v_i(S') - p(S')$$

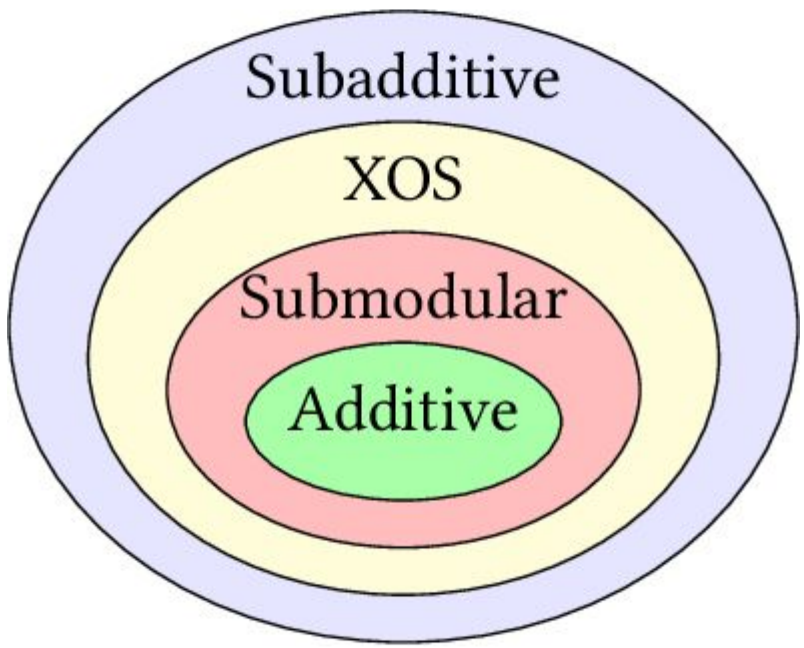
- $\boldsymbol{Rev}(S')$ for a bundle S' given a price vector i is defined as:

$$Rev(S') = \sum_{e \in S'} p(e)$$

- **Social Welfare** = $\sum_{i \in N} v_i(A_i) = \sum_{i \in N}(Utility(i) + Rev(A_i))$

Classes of valuation functions and assumptions

- This work presents great improvements in the **Subadditive** class and small improvements in the **XOS** class.
- **Subadditivity** translates to $v(S \cup T) \leq v(S) + v(T)$
- **Normalization** $v(\emptyset) = 0$
- **Monotonicity** $v(S) \leq v(T)\ \forall S \subseteq T \subseteq M$



BinarySearchMechanism

BinarySearchMechanism(N, M, ψ):

-

1. Let $B = \{0, 2^{-3\lceil \log m \rceil}\psi, 2^{1-3\lceil \log m \rceil}\psi, \dots, 2^{-1}\psi\}$.
2. Initialize $\boldsymbol{b}_e{}^{(1)}$ for all $e \in M$ to contain all elements of B ordered increasingly.
3. For every $i \in N$ draw r_i independently uniformly from $[\beta + 1]$, where r_i denotes the round in which bidder i participates. For any $\ell \in [\beta + 1]$, let $\mathrm{N}_\ell := \{i \mid r_i = \ell\}$ in the same order as N.
4. Select a uniformly random final allocation round $r^\star \in [\beta + 1]$.
5. For $\ell = 1$ to β rounds:
   a) For every item e, define the price $p_e{}^{(\ell)} = b_{e,\frac{k}{2}+1}{}^{(\ell)}$, where $k = {|B|}/{2^{\ell-1}}$ and $\boldsymbol{b}_e{}^{(\ell)} = (b_{e,1}{}^{(\ell)}, \dots, b_{e,k}{}^{(\ell)})$ is the vector of candidate prices for e.
   b) Run FixedPriceAuction(N_ℓ, M, $\boldsymbol{p}^{(\ell)}$) and let $A(\ell)$ be the allocation.
   c) For every item e, if e is allocated in $A(\ell)$, define remaining candidate prices $\boldsymbol{b}_e{}^{(\ell+1)} = (b_{e,\frac{k}{2}+1}{}^{(\ell)}, \dots, b_{e,k}{}^{(\ell)})$, and otherwise define $\boldsymbol{b}_e{}^{(\ell+1)} = (b_{e,1}{}^{(\ell)}, \dots, b_{e,\frac{k}{2}}{}^{(\ell)})$.
   d) If $r^\star = \ell$ then return $A^{(\ell)}$ as the final allocation.
6. Run FixedPriceAuction($\mathrm{N}_{\beta+1}, M, \boldsymbol{p}^{(\beta+1)}$), where $p_e{}^{(\beta+1)}$ in $\boldsymbol{p}^{(\beta+1)}$ is the unique price in $\boldsymbol{b}_e{}^{(\beta+1)}$, and return this as the final allocation.